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Derivation of Variable Annuity Dynamic Hedge Ratios with and without Dynamic Lapse

The purpose of this short note is threefold:

a) To outline an analytic Variable Annuity (VA) hedging ratio model[1] to use as an SR 26-2 style benchmark as part of a hedging model validation (see Reference [Federal Reserve System 2026]).
b) To extend the analytic hedging ratio model to incorporate surrender risk.
c) To comment on the impact the hedging model has on actuarial accounting (see Reference [NAIC 2018], SSAP 108).

1. Hedge Ratios without Surrender Risk

I employ straightforward multifactor replication theory to derive the hedge ratios quoted without proof in Reference [Ruez 2016]. It is useful to start with the standard dynamic hedging ratios that ignore lapse risk. Ruez captures equity S_t, interest rate r_t and variance V_t risk factors with the following physical $\mathbb{P}$-measure dynamics -

$$dV_t = \kappa^{(V)}\left(\theta^{(V)} - V_t\right)dt + \sigma^{(V)}\sqrt{V_t}dW_t^{(V)}$$

$$dr_t = \kappa^{(r)}\left(\theta^{(r)} - r_t\right)dt + \sigma^{(r)}\sqrt{r_t}\,dW_t^{(r)}$$

$$dS_t = (r_t + \mu)S_t dt + \sqrt{V_t}\,S_t dW_t^{(S)}$$

$$dC_t = r_t C_t dt$$

$$dW_t^{(r)}dW_t^{(S)} = dW_t^{(r)}dW_t^{(V)} = 0$$

The reader recognizes the equity process is Heston's stochastic volatility model and the short rate is driven by the Cox-Ingersoll-Ross model. The last equation above stipulates interest rates are independent of the equity process.

We assume the value of the VA liability, $V_t^{(\pi)}$, is strictly a function of the variance, short rate and equity process, *e.g.*, $V_t^{(\pi)} = V_t^{(\pi)}(V_t, r_t, S_t)$. Hence, by Ito's lemma the differential of the liability is -

$$dV_t^{(\pi)} = \frac{\partial V_t^{(\pi)}}{\partial S_t}dS_t + \frac{\partial V_t^{(\pi)}}{\partial V_t}dV_t + \frac{\partial V_t^{(\pi)}}{\partial r_t}dr_t + drift\ terms$$

[1] Based on Reference [Ruez 2016].

The self-financing replicating dynamic hedging portfolio consists of positions in the underlying equity fund of the policyholder's separate account (with value S_t), equity put options (with value O_t), zero-coupon bonds (with value Z_t), and a cash position (the money-market account, C_t). The $time - t$ value of the hedging portfolio is -

$$\Psi_t = \lambda_t^{(S)} S_t + \lambda_t^{(O)} O_t + \lambda_t^{(Z)} Z_t + \lambda_t^{(C)} C_t$$

Following the recipe of multifactor replication, we solve for the portfolio weights such that the hedge portfolio Ψ_t replicates the stochastic variation of the VA liability $V_t^{(\pi)}$.

Let's look at the partial derivatives of the replicating portfolio with respect to each of the key risk factors, V_t, S_t, r_t.

Only the put option has non-zero sensitivity to variance, $\frac{\partial O_t}{\partial V_t} \neq 0$. Hence,

$$\frac{\partial \Psi_t}{\partial V_t} = \lambda_t^{(O)} \ \frac{\partial O_t}{\partial V_t}$$

Now follow the replication recipe by setting this equal to the VA liability variance sensitivity

$$\lambda_t^{(O)} \ \frac{\partial O_t}{\partial V_t} = \frac{\partial V_t^{(\pi)}}{\partial V_t}$$

Solving for the option hedge ratio obtains

$$\boxed{\lambda_t^{(O)} = \frac{\frac{\partial V_t^{(\pi)}}{\partial V_t}}{\frac{\partial O_t}{\partial V_t}}}$$

Now we look at equity sensitivity of the replicating portfolio. Both the option and the equity positions have non-zero sensitivities as follows –

$$\frac{\partial \Psi_t}{\partial S_t} = \lambda^{(S)} \cdot 1 + \lambda_t^{(O)} \ \frac{\partial O_t}{\partial S_t}$$

Per the replication recipe, set this equal to the delta of the VA liability to obtain -

$$\lambda_t^{(S)} + \lambda_t^{(O)} \ \frac{\partial O_t}{\partial S_t} = \frac{\partial V_t^{(\pi)}}{\partial S_t}$$

Given that we obtained $\lambda^{(O)}$ above, we solve for $\lambda_t^{(S)}$ to obtain –

$$\lambda_t^{(S)} = \frac{\partial V_t^{(\pi)}}{\partial S_t} - \lambda_t^{(O)} \frac{\partial O_t}{\partial S_t}$$

Finally, we look at interest rate sensitivity of the replicating portfolio. Both the option and the zero-coupon bond provide the rho-sensitivity as follows –

$$\frac{\partial \Psi_t}{\partial r_t} = \lambda_t^{(Z)} \frac{\partial Z_t}{\partial r_t} + \lambda_t^{(O)} \frac{\partial O_t}{\partial r_t}$$

As before, we follow the replication recipe by setting this equal to the rate sensitivity of the VA liability –

$$\lambda_t^{(Z)} \frac{\partial Z_t}{\partial r_t} + \lambda_t^{(O)} \frac{\partial O_t}{\partial r_t} = \frac{\partial V_t^{(\pi)}}{\partial r_t}$$

Solving for the zero-coupon bond hedge-weight obtains

$$\lambda_t^{(Z)} = \frac{\frac{\partial V_t^{(\pi)}}{\partial r_t} - \lambda_t^{(O)} \frac{\partial O_t}{\partial r_t}}{\frac{\partial Z_t}{\partial r_t}}$$

The cash position is not a risk-hedging instrument but rather ensures the portfolio is self-financing at each rebalancing date. Given that Ψ_t is the current total portfolio value, and knowing the value invested in the three risky instruments, the cash residual must be –

$$\lambda_t^{(C)} = \Psi_t - \left[\lambda_t^{(S)} S_t + \lambda_t^{(O)} O_t + \lambda_t^{(Z)} Z_t\right]$$

We note that a key assumption is the independence of $W_t^{(r)}$ with respect to the equity Brownian motions $W_t^{(V)}$ and $W_t^{(S)}$. This assumption allows the $\rho - hedge$ to be cleanly separated from the Δ and $\mathcal{V}$ sensitivities.

2. Hedge Ratios with Dynamic Lapse Model

Via a straightforward application of the chain rule of differentiation, we can update the above formula for the hedge ratios to accommodate a lapse model. As was the case for the VA liability functional dependence, we assume the dynamic lapse is strictly a function of the variance, short rate and equity process, e.g., $s(t_i) = s(V_t, r_t, S_t)$. The result is the VA liability's value acquires new sensitivity channels through the lapse function.

In the following I define the set of all calculation times as $\mathcal{T} := \{t_i | \, i = 0, 1, \dots, N\}$. As defined by Ruez, let $s_i \coloneqq s(t_{i-1}, t_i)$ represent the fraction of policyholders who surrender their contracts at the end of the time interval $]t_{i-1}, t_i]$. In Section 1 the $s_i = 0$, whereas in the present section we allow for non-zero lapse.

Upon applying the chain rule, the augmented Δ for the VA liability becomes -

$$\left.\frac{\partial V_t^{(\pi)}}{\partial S_t}\right|_{dyn} = \underbrace{\left.\frac{\partial V_t^{(\pi)}}{\partial S_t}\right|_{s\ zero}}_{direct\ \Delta} + \underbrace{\sum_{i:t_i>t} \frac{\partial V_t^{(\pi)}}{\partial s_i}\frac{\partial s_i}{\partial S_t}}_{lapse-induced\ \Delta}$$

The augmented Vega for the VA liability becomes -

$$\left.\frac{\partial V_t^{(\pi)}}{\partial V_t}\right|_{dyn} = \underbrace{\left.\frac{\partial V_t^{(\pi)}}{\partial V_t}\right|_{s\ zero}}_{direct\ \text{Vega}} + \underbrace{\sum_{i:t_i>t} \frac{\partial V_t^{(\pi)}}{\partial s_i}\frac{\partial s_i}{\partial V_t}}_{lapse-induced\ \text{Vega}}$$

The augmented ρ for the VA liability becomes –

$$\left.\frac{\partial V_t^{(\pi)}}{\partial r_t}\right|_{dyn} = \underbrace{\left.\frac{\partial V_t^{(\pi)}}{\partial r_t}\right|_{s\ zero}}_{direct\ \rho} + \underbrace{\sum_{i:t_i>t} \frac{\partial V_t^{(\pi)}}{\partial s_i}\frac{\partial s_i}{\partial r_t}}_{lapse-induced\ \rho}$$

The same four instruments are employed for the replicating portfolio. The hedging ratios have the same form as in the no-lapse case, but use of the augmented Greeks calculated above –

$$\lambda_t^{(O,lapse)} = \frac{\left.\frac{\partial V_t^{(\pi)}}{\partial V_t}\right|_{dyn}}{\frac{\partial O_t}{\partial V_t}}$$

$$\lambda_t^{(S,lapse)} = \left.\frac{\partial V_t^{(\pi)}}{\partial S_t}\right|_{dyn} - \lambda_t^{(O)}\ \frac{\partial O_t}{\partial S_t}$$

$$\lambda_t^{(Z,lapse)} = \frac{\left.\frac{\partial V_t^{(\pi)}}{\partial r_t}\right|_{dyn} - \lambda_t^{(O)}\ \frac{\partial O_t}{\partial r_t}}{\frac{\partial Z_t}{\partial r_t}}$$

We note that lapse is also driven by policyholder irrationality, tax events, competitor products, and other non-market factors – these components of lapse cannot be replicated. These factors contribute to the hedging basis risk.

We can write a formula for the lapse sensitivity terms $\frac{\partial V_t^{(\pi)}}{\partial s_i}$ based on the risk-neutral valuation formula[2] for the VA liability given in Section 2.3 of Reference [Ruez 2016]. The value of the VA liability is -

$$V_t^{(\pi)} = E^{\mathbb{Q}}\left[\sum_{s\in\mathcal{T},s>t} \frac{C_t}{C_s}\left(G_s^{(P)} - G_s^{(C)}\right)\middle|\ \mathcal{F}_t\right]$$

This expectation is calculated with respect to the risk-neutral measure. $G_s^{(P)}$ are the VA guarantee payments and $G_s^{(C)}$ are the guarantee charges.

Differentiating the liability with respect to s_i yields –

$$\frac{\partial V_t^{(\pi)}}{\partial s_i} = E^{\mathbb{Q}}\left[\sum_{s\in\mathcal{T},s>t} \frac{C_t}{C_s}\frac{\partial}{\partial s_i}\left(G_s^{(P)} - G_s^{(C)}\right)\middle|\ \mathcal{F}_t\right]$$

Because lapses reduce the insurer's net liability, I expect this partial derivative to be negative. However, the surrender fees and whether the lapse event in in or out of the money (depending on the rationality of the policyholder) also contribute to the sign of this sensitivity.

[2] The numeraire asset is the money-market account, C_t.

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3. Impact on Hedge Effectiveness Testing

One final observation relating to the practice of VA hedging and its actuarial accounting treatment. SSAP No. 108 (Reference [NAIC 2018] below) was specifically created because the Greek-based dynamic hedging programs for VA guarantees — of exactly the type modeled in the Ruez paper — cannot pass SSAP No. 86's (see Reference [NAIC 2022-SSAP 86] below) standard hedge effectiveness tests (the 80–125% offset requirement). SSAP No. 108 instead compares fair value fluctuations of the derivatives to changes in the VM-21 reserve liability and allows excess gains/losses to be deferred and amortized over the Macaulay duration of the guarantee benefit cash flows, not to exceed 10 years. Reference [NAIC 2018], Issue Paper No. 159 explains this rationale in full.

The SSAP 108 framework has a direct correspondence to the academic hedging model in the Ruez paper discussed in Sections 1 and 2 above. The sensitivity $\frac{\partial V_t^{(\pi)}}{\partial r_t}$ in Ruez's hedge ratio$\lambda_t^{(Z)}$ is precisely the quantity that drives both the SSAP 108's prospective effectiveness assessment and the deferred asset/liability calculation. The "augmented Rho" derived when lapse risk is incorporated - $\left.\frac{\partial V_t^{(\pi)}}{\partial r_t}\right|_{dyn}$ - is the correct input to use when policies exhibit dynamic surrender behavior, replacing the static Rho in both the hedge ratio formula and the SSAP 108 effectiveness methodology.

References

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[Ruez 2016] Ruez, Frederick, Variable Annuities with Guaranteed Lifetime Withdrawal Benefits: An Analysis of Risk-Based Capital Requirements, Working Paper 2016.

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